

# Comments on “Approximate Characterizations for the Gaussian Source Broadcast Distortion Region”

Lei Yu, Houqiang Li, *Senior Member, IEEE*, and Weiping Li, *Fellow, IEEE*

## Abstract

Recently, Tian *et al.* [1] considered joint source-channel coding of transmitting a Gaussian source over  $K$ -user Gaussian broadcast channel, and derived an outer bound on the admissible distortion region. In [1], they stated “due to its nonlinear form, it appears difficult to determine whether it is always looser than the trivial outer bound in all distortion regimes with bandwidth compression”. However, in this correspondence we solve this problem and prove that for the bandwidth expansion case ( $K \geq 2$ ), this outer bound is strictly tighter than the trivial outer bound with each user being optimal in the point-to-point setting; while for the bandwidth compression or bandwidth match case, this outer bound actually degenerates to the trivial outer bound. Therefore, our results imply that on one hand, the outer bound given in [1] is nontrivial only for Gaussian broadcast communication ( $K \geq 2$ ) with bandwidth expansion; on the other hand, unfortunately, no nontrivial outer bound exists so far for Gaussian broadcast communication ( $K \geq 2$ ) with bandwidth compression.

## Index Terms

Outer bound, Gaussian source, Gaussian broadcast channel, joint source-channel coding (JSCC), squared error distortion, nontrivial bound.

## I. INTRODUCTION AND PRELIMINARIES

Recently, Tian *et al.* [1] considered joint source-channel coding (JSCC) of transmitting a Gaussian source over  $K$ -user Gaussian broadcast channel, and derived an outer bound on the admissible distortion

The authors are all with the Department of Electronic Engineering and Information Science, University of Science and Technology of China, Hefei, China (e-mail: {yulei,lihq,wpli}@ustc.edu.cn).

region. For  $K = 2$  case, the properties of the outer bound were thoroughly investigated by Reznic *et al.* in [2], and in this case, Tian *et al.* [1] and Reznic *et al.* [2] clarified that in certain regimes, this outer bound in fact degenerates for the case of bandwidth compression, and it is looser than the trivial outer bound with each user being optimal in the point-to-point setting. However, the nonlinear form of the bound was cited as the main difficulty preventing a direct determination whether this outer bound is always looser than the trivial outer bound in all distortion regimes with bandwidth compression. Although [1] states “this outer bound always holds whether the bandwidth is expanded or compressed”, in this correspondence we prove that for the bandwidth expansion case (with  $K \geq 2$ ), this outer bound is strictly tighter than the trivial outer bound; while for the bandwidth compression or bandwidth match case, this outer bound actually degenerates to the trivial outer bound. It means that on one hand, for Gaussian broadcast communication ( $K \geq 2$ ) with bandwidth compression, no nontrivial outer bound exists so far; on the other hand, for Gaussian broadcast communication ( $K \geq 2$ ) with bandwidth expansion, the outer bound given in [1] is nontrivial.

The correspondence is organized as follows. In Section II, we revisit the outer bound on the admissible distortion region given in [1], and then prove it to be trivial for bandwidth compression and bandwidth match cases, and nontrivial for bandwidth expansion case. In Section III, we give the concluding remarks.

### Preliminaries

The Minkowski inequality given in the following lemma plays an important role in proving our results.

**Lemma 1** (Minkowski inequality). [3] *For real numbers or infinity  $0 \leq x_i, y_i \leq +\infty$ ,  $i = 1, \dots, n$ , and for  $0 < p < 1$ , it holds that*

$$\left( \sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^n y_i^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^n (x_i + y_i)^p \right)^{\frac{1}{p}}. \quad (1)$$

*Moreover, for  $p > 1$ , the inequality is reversed. In each case equality holds if and only if the sequences  $\{x_i\}$  and  $\{y_i\}$  are positively linearly dependent (i.e.,  $y_i = \lambda x_i$ ,  $i = 1, \dots, n$  for some  $\lambda \geq 0$  or  $x_i = 0$ ,  $i = 1, \dots, n$ ), or there exists some  $x_i$  or  $y_i$  equal to  $+\infty$ .*

## II. MAIN RESULTS

Consider the problem of broadcasting a Gaussian source  $S$  with unit-variance, i.e.,  $N_S = 1$ , over a  $K$ -user Gaussian broadcast channel  $Y_k = X + Z_k$ ,  $k = 1, \dots, K$  with channel noise variances  $N_1 > N_2 > \dots > N_K > 0$  and transmitting power  $P > 0$ . The encoder maps a source sample block of length

$m$  into a channel input block of length  $n$ , and each decoder maps the corresponding channel output block of length  $n$  into a source reconstruction block of length  $m$ . The bandwidth mismatch factor  $b$  is defined as  $b = \frac{n}{m}$ . For an  $(m, n, P, d_1, d_2, \dots, d_K)$  Gaussian source-channel broadcast code with encoding function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  and decoding function  $g_k : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $k = 1, \dots, K$ , such that  $\frac{1}{n} \sum_{i=1}^n \mathbb{E} (X(i))^2 \leq P$ , the induced distortions are defined as  $d_k = \frac{1}{m} \sum_{i=1}^m \mathbb{E} \left( S(i) - \hat{S}_k(i) \right)^2$ , where  $\hat{S}_k^m = g_k(Y_k^n)$  is the source reconstruction at receiver  $k$ . A distortion tuple  $(D_1, D_2, \dots, D_K)$  is said to be achievable under power constraint  $P$  and bandwidth mismatch factor  $b$ , if for any  $\epsilon > 0$  and sufficiently large  $m$ , there exist an integer  $n \leq mb$  and a Gaussian source-channel broadcast code  $(m, n, P, d_1, d_2, \dots, d_K)$  such that  $d_k \leq D_k + \epsilon, k = 1, \dots, K$ . On this problem, one outer bound is derived in [1] and shown as follows.

**Theorem 1.** [1, Thm. 2] Let  $0 = \tau_K \leq \tau_{K-1} \leq \dots \leq \tau_1 \leq +\infty^1$  be any nonnegative real values or infinity. If distortion tuple  $(D_1, D_2, \dots, D_K)$  is achievable, then

$$\sum_{k=1}^K \Delta N_k \left[ \frac{(1 + \tau_k) \prod_{j=2}^k (D_j + \tau_{j-1})}{\prod_{j=1}^k (D_j + \tau_j)} \right]^{\frac{1}{b}} \leq P + N_1, \quad (2)$$

where  $\Delta N_k = N_k - N_{k+1}, 1 \leq k \leq K-1$  and  $\Delta N_K = N_K$ .

In addition, according to cut-set bound, each receiver cannot achieve a lower distortion than the optimal one in the point-to-point setting, i.e.,

$$D_k \geq D_k^*, k = 1, \dots, K, \quad (3)$$

with

$$D_k^* = \left( \frac{N_k}{P + N_k} \right)^b, k = 1, \dots, K. \quad (4)$$

This point-to-point outer bound is referred to as *trivial outer bound*.

For any  $k, 1 \leq k \leq K$ , let  $\tau_K = \dots = \tau_k = 0$  and  $\tau_{k-1} = \dots = \tau_1 = +\infty$ , then the inequality (2) reduces to (3). Hence we get the following proposition.

**Proposition 1.** For any distortion tuple  $(D_1, D_2, \dots, D_K)$ , if it satisfies the inequality (2) for any  $0 = \tau_K \leq \tau_{K-1} \leq \dots \leq \tau_1 \leq +\infty$ , then  $D_k \geq D_k^*, k = 1, \dots, K$ .

This proposition implies that the outer bound in Theorem 1 is tighter than or as tight as the trivial one. However, under what conditions it is strictly tighter than the trivial outer bound, is still unknown.

<sup>1</sup>For ease of analysis, different from [1, Thm. 2], here we allow  $\tau_k$ 's to be infinity. This makes no difference to the outer bound, since the inequality (2) is non-strict.

Next we will address this problem. First, we consider the bandwidth compression case, and show that for this case the trivial outer bound  $(D_1^*, D_2^*, \dots, D_K^*)$  satisfies the necessity given in Theorem 1, i.e.,  $(D_1^*, D_2^*, \dots, D_K^*)$  belongs to the outer bound region given by Theorem 1.

**Theorem 2.** *Let  $0 = \tau_K \leq \tau_{K-1} \leq \dots \leq \tau_1 \leq +\infty$  be any nonnegative real values or infinity. If  $0 < b < 1$ , then the distortion tuple  $(D_1^*, D_2^*, \dots, D_K^*)$  satisfies the inequality (2).*

*Proof:* We adopt mathematical induction to prove Theorem 2.

**Step 1:** For  $K = 1$ , we have

$$\begin{aligned} & \sum_{k=1}^1 \Delta N_k \left[ \frac{(1 + \tau_k) \prod_{j=2}^k (D_j^* + \tau_{j-1})}{\prod_{j=1}^k (D_j^* + \tau_j)} \right]^{\frac{1}{b}} \\ &= \Delta N_1 \left( \frac{1}{D_1^*} \right)^{\frac{1}{b}} \end{aligned} \tag{5}$$

$$= P + N_1. \tag{6}$$

**Step 2:** For  $K = 2$ , we have

$$\begin{aligned} & \sum_{k=1}^2 \Delta N_k \left[ \frac{(1 + \tau_k) \prod_{j=2}^k (D_j^* + \tau_{j-1})}{\prod_{j=1}^k (D_j^* + \tau_j)} \right]^{\frac{1}{b}} \\ &= \Delta N_1 \left( \frac{1 + \tau_1}{D_1^* + \tau_1} \right)^{\frac{1}{b}} + \Delta N_2 \left( \frac{D_2^* + \tau_1}{D_2^* (D_1^* + \tau_1)} \right)^{\frac{1}{b}} \end{aligned} \tag{7}$$

$$= \Delta N_1 \left( \frac{1 + \tau_1}{D_1^* + \tau_1} \right)^{\frac{1}{b}} + (P + N_2) \left( \frac{D_2^* + \tau_1}{D_1^* + \tau_1} \right)^{\frac{1}{b}} \tag{8}$$

$$= (D_1^* + \tau_1)^{-\frac{1}{b}} \left[ \Delta N_1 (1 + \tau_1)^{\frac{1}{b}} + (P + N_2) (D_2^* + \tau_1)^{\frac{1}{b}} \right] \tag{9}$$

$$= (D_1^* + \tau_1)^{-\frac{1}{b}} \left\{ \left[ (\Delta N_1)^b + (\Delta N_1)^b \tau_1 \right]^{\frac{1}{b}} + \left[ (N_2)^b + (P + N_2)^b \tau_1 \right]^{\frac{1}{b}} \right\} \tag{10}$$

$$\leq (D_1^* + \tau_1)^{-\frac{1}{b}} \left[ (N_2 + \Delta N_1)^b + (P + N_2 + \Delta N_1)^b \tau_1 \right]^{\frac{1}{b}} \tag{11}$$

$$= (D_1^* + \tau_1)^{-\frac{1}{b}} \left[ (N_1)^b + (P + N_1)^b \tau_1 \right]^{\frac{1}{b}} \tag{12}$$

$$= P + N_1, \tag{13}$$

where (11) follows from the Minkowski inequality (Lemma 1).

**Step 3:** Assume Theorem 2 holds for  $K = K' \geq 2$ . Then for  $K = K' + 1$ , we have

$$\begin{aligned} & \sum_{k=1}^{K'+1} \Delta N_k \left[ \frac{(1 + \tau_k) \prod_{j=2}^k (D_j^* + \tau_{j-1})}{\prod_{j=1}^k (D_j^* + \tau_j)} \right]^{\frac{1}{b}} \\ &= \Delta N_1 \left( \frac{1 + \tau_1}{D_1^* + \tau_1} \right)^{\frac{1}{b}} + \sum_{k=2}^{K'+1} \Delta N_k \left[ \frac{(1 + \tau_k) \prod_{j=2}^k (D_j^* + \tau_{j-1})}{\prod_{j=1}^k (D_j^* + \tau_j)} \right]^{\frac{1}{b}} \end{aligned} \quad (14)$$

$$= \Delta N_1 \left( \frac{1 + \tau_1}{D_1^* + \tau_1} \right)^{\frac{1}{b}} + \left( \frac{D_2^* + \tau_1}{D_1^* + \tau_1} \right)^{\frac{1}{b}} \sum_{k=2}^{K'+1} \Delta N_k \left[ \frac{(1 + \tau_k) \prod_{j=3}^k (D_j^* + \tau_{j-1})}{\prod_{j=2}^k (D_j^* + \tau_j)} \right]^{\frac{1}{b}} \quad (15)$$

$$\leq \Delta N_1 \left( \frac{1 + \tau_1}{D_1^* + \tau_1} \right)^{\frac{1}{b}} + \left( \frac{D_2^* + \tau_1}{D_1^* + \tau_1} \right)^{\frac{1}{b}} (P + N_2) \quad (16)$$

$$\leq P + N_1, \quad (17)$$

where (16) follows from the assumption Theorem 2 holds for  $K = K'$ , and (17) follows from the formulas (9)-(13). From (17), Theorem 2 holds for  $K = K' + 1$ .

By combining Steps 1-3, Theorem 2 holds for any  $K \geq 1$ . This completes the proof.  $\blacksquare$

Combine Proposition 1 and Theorem 2, then we get the following corollary.

**Corollary 1.** Assume  $0 < b < 1$ . Then for any distortion tuple  $(D_1, D_2, \dots, D_K)$ , it satisfies the inequality (2) for any  $0 = \tau_K \leq \tau_{K-1} \leq \dots \leq \tau_1 \leq +\infty$ , if and only if  $D_k \geq D_k^*, k = 1, \dots, K$ .

*Remark 1.* Corollary 1 implies that for the bandwidth compression case, the outer bound given in Theorem 1 degenerates to the trivial one.

*Proof:* Actually, the “only if” part directly follows from Proposition 1, hence we only need prove the “if” part.

Define a function  $g(D_1, \dots, D_K)$  as

$$\begin{aligned} & g(D_1, \dots, D_K) \\ & \triangleq \sum_{k=1}^K \Delta N_k \left[ \frac{(1 + \tau_k) \prod_{j=2}^k (D_j + \tau_{j-1})}{\prod_{j=1}^k (D_j + \tau_j)} \right]^{\frac{1}{b}} \end{aligned} \quad (18)$$

$$= \sum_{k=1}^K \Delta N_k \left[ \left( \frac{1 + \tau_k}{D_1 + \tau_1} \right) \prod_{j=2}^k \left( \frac{D_j + \tau_{j-1}}{D_j + \tau_j} \right) \right]^{\frac{1}{b}}. \quad (19)$$

Obviously,  $g(D_1, \dots, D_K)$  is monotonically nonincreasing in  $D_k$ , i.e.,  $\frac{\partial g}{\partial D_k} \leq 0$ . Therefore, we have  $g(D_1, \dots, D_K) \leq g(D_1^*, \dots, D_K^*)$ . Combining it with Theorem 2, i.e.,  $g(D_1^*, \dots, D_K^*) \leq P + N_1$ , we

have  $g(D_1, \dots, D_K) \leq P + N_1$ . It implies the “if” part holds.  $\blacksquare$

We can also prove the corresponding results for the bandwidth expansion case and bandwidth match case.

**Theorem 3.** *Let  $0 = \tau_K \leq \tau_{K-1} \leq \dots \leq \tau_1 \leq +\infty$  be any nonnegative real values or infinity. If  $b > 1$ , then the distortion tuple  $(D_1^*, D_2^*, \dots, D_K^*)$  satisfies*

$$\sum_{k=1}^K \Delta N_k \left[ \frac{(1 + \tau_k) \prod_{j=2}^k (D_j^* + \tau_{j-1})}{\prod_{j=1}^k (D_j^* + \tau_j)} \right]^{\frac{1}{b}} \geq P + N_1. \quad (20)$$

Moreover, if  $b > 1$ ,  $K \geq 2$  and there exists at least one  $\tau_k$ ,  $1 \leq k \leq K$  such that  $0 < \tau_k < +\infty$ , then the strict inequality holds in (20).

*Remark 2.* Theorem 3 implies that for the broadcast with bandwidth expansion and at least two receivers, the outer bound given in Theorem 1 is (strictly) nontrivial, i.e., it is strictly tighter than the trivial outer bound.

**Theorem 4.** *Let  $0 = \tau_K \leq \tau_{K-1} \leq \dots \leq \tau_1 \leq +\infty$  be any nonnegative real values or infinity. If  $b = 1$ , then the distortion tuple  $(D_1^*, D_2^*, \dots, D_K^*)$  satisfies*

$$\sum_{k=1}^K \Delta N_k \left[ \frac{(1 + \tau_k) \prod_{j=2}^k (D_j^* + \tau_{j-1})}{\prod_{j=1}^k (D_j^* + \tau_j)} \right]^{\frac{1}{b}} = P + N_1. \quad (21)$$

*Remark 3.* Theorem 4 implies that for the bandwidth match case,  $(D_1^*, D_2^*, \dots, D_K^*)$  satisfies the equality (21) for any  $0 = \tau_K \leq \tau_{K-1} \leq \dots \leq \tau_1 \leq +\infty$ . In addition, analog coding could achieve  $(D_1^*, D_2^*, \dots, D_K^*)$ , hence for this case, the outer bound given in Theorem 1 is tight. Besides, equality (21) can be also obtained by examining all inequalities used to derived the outer bound in [1].

The proofs of Theorem 3 and Theorem 4 are similar to that of Theorem 2, and hence omitted here.

### III. CONCLUDING REMARKS

In this correspondence, we revisited the outer bound on the admissible distortion region for Gaussian broadcast communication derived in [1], and then proved it to be tighter than the trivial bound for bandwidth expansion case, and degenerate into the trivial one for bandwidth match and bandwidth compression cases. It means that on one hand, for Gaussian broadcast communication ( $K \geq 2$ ) with bandwidth compression, no nontrivial outer bound exists so far; on the other hand, for Gaussian broadcast communication ( $K \geq 2$ ) with bandwidth expansion, the outer bound given in [1] is nontrivial. Our results lead to a better

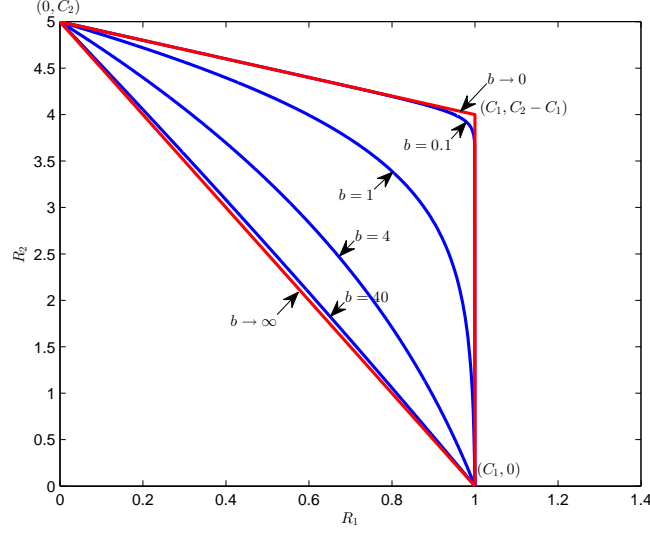


Fig. 1. Illustration of the fact that the capacity region for Gaussian broadcast channel with transmitting power  $P$  and noise variances  $N_1$  and  $N_2$  shrinks as the bandwidth  $b$  increases under the point-to-point capacity constraint for each receiver, i.e.,  $\mathcal{C}_b(P, N_1, N_2) \triangleq \{(R_1, R_2) : 0 \leq R_1 \leq \frac{b}{2} \log \frac{P+N_1}{\alpha P+N_1}, 0 \leq R_2 \leq \frac{b}{2} \log \frac{\alpha P+N_2}{N_1}, 0 \leq \alpha \leq 1\}$  shrinks as  $b$  increases under  $\frac{b}{2} \log(1 + \frac{P}{N_1}) = C_1$  and  $\frac{b}{2} \log(1 + \frac{P}{N_2}) = C_2$ . For this figure,  $C_1 = 1$  and  $C_2 = 5$ .

understanding of this outer bound, particularly, its relation with the trivial one. In [1], the outer bound is derived by introducing a set of auxiliary random variables (or remote sources). The essence of this proof method lies in that the conditional probability distribution  $p_{\hat{S}_1, \hat{S}_2, \dots, \hat{S}_K | S}$  can be considered as a virtual broadcast channel (induced by the source and the reconstructions) realized over the physical broadcast channel  $p_{Y_1, Y_2, \dots, Y_K | X}$ , hence the capacity region of such virtual broadcast channel should be contained inside that of the physical broadcast channel (see [4], [5], [6]). On the other hand, it can be verified the outer bound given in [1] is just the necessary condition  $\mathcal{C}(N_S, D_1, \dots, D_K) \subseteq b\mathcal{C}(P, N_1, \dots, N_K)$ , where  $\mathcal{C}(N_S, D_1, \dots, D_K)$  is the capacity region of the virtual Gaussian broadcast channel with transmitting power  $N_S$  and channel noise variances  $\frac{N_S D_k}{N_S - D_k}, 1 \leq k \leq K$  with  $N_S = 1$  as assumed previously, and  $\mathcal{C}(P, N_1, \dots, N_K)$  is the capacity region of the physical Gaussian broadcast channel with transmitting power  $P$  and channel noise variances  $N_k, 1 \leq k \leq K$ . Therefore, determining whether  $(D_1^*, D_2^*, \dots, D_K^*)$  belongs to the outer bound region given in [1] is equivalent to determining whether  $\mathcal{C}(N_S, D_1^*, \dots, D_K^*) \subseteq b\mathcal{C}(P, N_1, \dots, N_K)$ , where  $\mathcal{C}(N_S, D_1^*, \dots, D_K^*)$  is  $\mathcal{C}(N_S, D_1, \dots, D_K)$  with  $D_k = D_k^*, 1 \leq k \leq K$ . Note that for  $D_k = D_k^*, 1 \leq k \leq K$  case, the virtual broadcast channel and the physical broadcast channel have different bandwidth (the bandwidth ratio is  $b$ ) but the same point-to-point capacity for each receiver.

Combine these with our results, then we have  $\mathcal{C}(N_S, D_1^*, \dots, D_K^*) \subseteq b\mathcal{C}(P, N_1, \dots, N_K)$  for  $b < 1$ ,  $\mathcal{C}(N_S, D_1^*, \dots, D_K^*) = b\mathcal{C}(P, N_1, \dots, N_K)$  for  $b = 1$ , and  $\mathcal{C}(N_S, D_1^*, \dots, D_K^*) \supseteq b\mathcal{C}(P, N_1, \dots, N_K)$  for  $b > 1$ , which further means that the capacity region for Gaussian broadcast channel shrinks as the bandwidth increases under the point-to-point capacity constraint for each receiver as shown in Fig. 1. This provides an intuitive explanation why the outer bound in [1] is nontrivial only for bandwidth expansion.

In addition, from the derivation of the outer bound in [1], it seems that only the one-to-one (continuous) linear analog coding could achieve this outer bound. However, for bandwidth mismatch case, the one-to-one continuous mapping does not exist [7], [8], hence we conjecture that for both bandwidth compression case or expansion case, the outer bound probably cannot be achieved by any source-channel code.

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